Linear stochastic dynamics with nonlinear fractal properties

Didier Sornette[1,2]

[1] *Laboratoire de Physique de la Matière Condensée, CNRS URA190*

Université des Sciences, B.P. 70, Parc Valrose, 06108 Nice Cedex 2, France

[2] *Department of Earth and Space Science*

and Institute of Geophysics and Planetary Physics

University of California, Los Angeles, California 90095, USA

Abstract Stochastic processes with multiplicative noise have been studied independently in several different contexts over the past decades. We focus on the regime, found for a generic set of control parameters, in which stochastic processes with multiplicative noise produce intermittency of a special kind, characterized by a power law probability density distribution. We present a review of applications, highlight the common physical mechanism and summarize the main known results. The distribution and statistical properties of the duration of intermittent bursts are also characterized in details.

1 Introduction

Consider the number of fish X_t in a lake in the t-th year. Let X_{t+1} be related to the population X_t through

$$X_{t+1} = aX_t + b, \tag{1}$$

where $a = a(t) > 0$ and $b = b(t) > 0$ are drawn from some probability density function (pdf). The growth rate a depends on the rate of reproduction and the depletion rate due to fishing or predation, as well as on environmental conditions, and is therefore a variable quantity. The quantity b describes the input due to restocking from an external source such as a fish hatchery in artificial cases, or from migration from adjoining reservoirs in natural cases; b can but need not be constant. At first sight, it seems that the linear model (1) is so simple that it does not deserve a careful theoretical investigation. However it turns out that this is not the case: see, for example, a rather complicated mathematical analysis of the problem in [1]. It turns out that model (1) exhibits an unusual type of intermittency with a power law pdf of the variable X_t, for a large range of pdf's for a and b. As we show, the non-trivial properties of this simple model (1) come from the competition between the multiplicative and additive terms.

If $b = 0$ then (1) describes a simple multiplicative process with a distribution which is log-normal in its central part. If $a = 0$, then $X_t = b$. If $(a(t), b(t))$ are the positive constants (a, b), then X_t converges in the limit of large t to the attractor at the stable fixed point $\frac{b}{1-a}$ for $a < 1$ and diverges if $a > 1$. In the continuum limit of large t, assuming differentiability, the difference equation becomes a first order differential equation whose solution is $X(t) = \frac{b}{1-a} + Ce^{(a-1)t}$. The convergence (divergence) to (from) the fixed point is exponential with the rate $(a-1)^{-1}$ per unit time.

In the case of divergence, limits on growth have often been introduced through the mechanism of nonlinear saturation, in the absence of external restocking. Probably the most studied case of such nonlinearity is the logistic equation [2, 3], which exhibits a panoply of dynamical behavior that ranges from bifurcations via a period doubling sequence, to deterministic chaos, among other properties, as well as certain universal properties [4]. However, because of the saturation, the nonlinear dynamics is not able

to simulate the observations of self-similar behavior that extend to large values of the variable $X(t)$.

Although the model (1) appears to be an apparently innocuous AR(1) (autoregressive of order 1) process, the *random and positive* multiplicative coefficient a lead to non-trivial intermittent behavior for a large class of distributions for a. Standard stochastic population models [5] are often described by (1) with *fixed* coefficients. The intermittency arises because of the multiplicative noise structure, taken together with the additive source, and are not a property of usual AR processes with fixed coefficients [6]. Earlier investigations of auto-regressive models with random coefficients [7] have focused on values of the parameters wherein X_t exhibits a stationary variance, and thus exclude the highly intermittent regime we describe here which is characterized by power law distributions with finite and infinite variance.

The formal solution of (1) for $t \geq 1$ can be obtained explicitly

$$X_t = (\prod_{l=0}^{N-1} a(l))X_0 + \sum_{l=0}^{N-1} b(l) \prod_{m=l+1}^{N-1} a(m) \quad , \tag{2}$$

where, to deal with $l = N-1$, we define $\prod_{m=N}^{N-1} a(m) \equiv 1$. Because of the successive multiplicative operations on a in the iteration of X_t, it is the average logarithmic growth rate $\langle \log a \rangle$ that controls whether the population grows or dies. If the average logarithmic growth rate $\langle \log a \rangle$ is negative due to overfishing and other environmental influences (in the example), then the population must ultimately disappear if there is no restocking through the b term. On the other hand, X_t diverges with increasing time if $\langle \log a \rangle \geq 0$. Thus, the process decays or grows depending on whether the rate $t_c^{-1} = \langle \log a \rangle$ is negative or positive. In the case of decay, the introduction of new stock b has a persistent influence on X_t over the correlation time $|t_c|$. In absence of nonlinear saturation mechanisms, we consider first the regime where $\langle \log a \rangle$ is negative. From (2), it is clear that the Lyapunov exponent is nothing but $\langle \log a \rangle$. Since most of the interesting regime occurs for negative $\langle \log a \rangle$, this would give again the impression of a trivial behavior, if the additive term was not present.

Model (1) has similarities with the on-off intermittency described in [8]. In particular, the addition of additive noise on top of the multiplicative random forcing of the bifurcation parameter can be described by a biased random walk repelled from the origin in both problems [8, 9]. The same mechanism of intermittent multiplicative

amplifications applies to both problems. However, the main difference is that the nonlinear saturation in the models studied in [8] prevents the appearence of large excursions and the power law tail does not exist. Most of the analysis of on-off intermittency has been devoted to the unstable regime, corresponding in our notations to $\langle \log a \rangle > 0$, and to the calculation of the distribution of probability for a laminar phase of a given length. In contrast, we stress that model (1) has its most interesting regime for the "stable" case $\langle \log a \rangle < 0$ and we discuss the pdf of the variable itself.

Ref.([9]) presents figures of typical evolutionary sequence X_t for the uniform distributions $0.48 \leq a \leq 1.48$ and $0 \leq b \leq 1$. In this case $\langle \log a \rangle = -0.06747$ and $\langle a \rangle = 0.98$. For these parameters, $\langle X_t \rangle = \frac{\langle b \rangle}{1-\langle a \rangle} = 25$. Most of the time, X_t is significantly less than its average, while rare intermittent bursts propel it to very large values. Qualitatively, this behavior arises because of the occurrence of sequences of successive individual values of a that are larger than 1 which, although rare, are always bound to occur in this random process. The persistence of the temporal behavior has a decay which is $\exp[\langle \log a \rangle \tau]$ on the average, with correlation time $1/|\langle \log a \rangle| = 14.8$ of the influence of restocking by the amount $b(t-\tau)$ at a time τ in the past on the present value X_t. The distribution of X_t from a numerical realization with the properties above gives an histogram characterized by a power law tail

$$P(X) \sim X^{-(1+\mu)} \quad , \tag{3}$$

and a rolloff at smaller values of X_t, which is a result that is mandated by the fact that as $X(t) \rightarrow 0$ the process is dominated by the injection of new stock $0 < b < 1$, so that the population is repelled from a zero value.

In the next section, we discuss a series of applications such as population dynamics with external sources, epidemics where we provide a rationalization for observed power law data, finance and insurance, immigration and investment portfolios where we generalize (1) to expanding processes, and the internet. In section 3, we review and synthesize useful results. First, we recall a generalization of equation (1) to general contractive maps with intermittent bursts with a repulsion from the origin. We then explain how the power law pdf can be calculated. We connect these processes to auto-catalytic equations that were found to also exhibit a power law pdf (for the same reason), but whose treatment using the Fokker-Planck formalism was limited to

gaussian (multiplicative) noise. We briefly mention log-periodicity that was recently found as corrections to the leading power law pdf. We exhibit the relationship with intermittency in nonlinear dynamical system and discuss the multiaffine structure of the corresponding time series. We end this section by reviewing and extending known results on the durations of the intermittent bursts and their distribution. We also summarize our knowledge on the extremes of the random variable X_t. Section 4 proposes a nonlinear extension to the process (1) that includes the optimization of the restocking term b to develop an optimal strategy for population control. Section 5 concludes.

2 Domains of application

2.1 Population dynamics with external sources

Equation (1) is the simplest discrete map with external source one can think of. Notwithstanding the simplicity of its formulation, it exhibits a rich phenomenology, suitable to apply to population dynamics. Beyond the example of the fish population with restocking, agriculture provides another example, in which one wishes to model the annual fluctuations of the size of a crop in the case where, on average, the crop that is harvested leaves a seed crop residue that is less than that needed to fully reseed for the following year; in this case the variable b represents the seed purchased or otherwise obtained from outside sources.

Similarly, the large biological variability of phytoplankton blooms in shallow coastal ecosystems, such as estuaries, lagoons, bays and tidal rivers, can also be explained by a population budget given by equation (1) [10] where a accounts for the difference between growth rate and loss rate to respiration, to pelagic and benthic grazing, and to exchanges of biomass vertically between the top of the sediments and the overlying water column and horizontally due to advective and diffusion transport; the b term models the effects of injection of new individuals across the boundaries of a given ecosystem.

The results for (1) might explain the spontaneous large variability of such systems that are observed. Among still other problems, it would be interesting to explore whether the variability of long term geophysical time series, such as river discharge or

rainfall, which have been proposed in [11] to be self-affine (see (30)), can be framed in these simple terms or within their multi-dimensional generalizations described below.

In population dynamics, empirical observations show that adjacent generations tend to be less different than distant ones, a problem that has plagued standard nonlinear dynamical models [12, 13]. Models (1) and (18) do not share this difficulty and agree with this empirical fact. This suggests that intermittent multiplicative noise might be a useful ingredient for future modelling of biological populations.

2.2 Epidemics

The number S of new cases of people infected with measles in isolated islands in the North Atlantic per epidemic event is observed to have a power law distribution with exponent about 0.3 (called β in [14]):

$$P(S)dS \sim S^{-(1+\beta)}\ dS\ . \tag{4}$$

The durations of epidemic events also have a power law distribution $P(\tau)$ with an exponent $\gamma \approx 0.8$. Equation (1) is a simple model of multiplicative growth that accounts for the observation of the power law distributions. The multiplicative term in (1) represents the spread though personal infectious contact and the additive term is representative of the introduction of new strains by agents from outside. Using this insight, we can predict $\gamma = 1$ from the following self-consistent argument.

Let X_t be the number of new cases between $t-1$ and t in (1) which is distributed according to a power law distribution $P(X) \sim X^{-(1+\mu)}$, as given in (3) and (22), with exponent μ. Notice that μ is a priori different from β as μ describes the pdf of fluctuations within an epidemic event while β describes the pdf of the total size of that event. The total number of cases in an epidemic event of duration τ is

$$S_\tau = \sum_0^\tau X_t\ . \tag{5}$$

For $\mu < 1$, we find

$$S_\tau \sim \tau^{1/\mu}\ . \tag{6}$$

From the conservation of probability under a change of variable, we get

$$P(\tau)d\tau \sim \tau^{-(1+\beta/\mu)}\ d\tau\ , \tag{7}$$

i.e.

$$\gamma \equiv \frac{\beta}{\mu} \ . \tag{8}$$

In section 3.8 below, eq.(41) finds, from a random walk argument, that $\gamma = 1/2$ in the uncorrelated case. If we include correlations by an argument of self-similarity of epidemics within epidemics, i.e. the burst of new cases within a given epidemic event is a small epidemic event in itself (we define an epidemic event to be the sudden awakening from a relatively dormant state), we get $\beta = \mu$ and thus $\gamma = 1$. This seems consistent with the observation $\gamma \approx 0.8$ within statistical estimates of the error in the exponent in power law distributions [15].

2.3 Finance and insurance applications with relation to ARCH(1) process

The relation between (1) and finance has been pointed out in several papers (see for instance [16]. Suppose you invest $b(t)$ at time 0,1,2,... in a bond with interest time-dependent interest rate r_t. The accumulated value X_t at time t of the interest payments made at times 0,..., t, assuming $Y_0 = 1$ is given by the equation

$$X_{t+1} = b(t) + r_t X_t \ . \tag{9}$$

The inverse problem to accumulation is discounting. Suppose payments $b(i)$ are made at times $0, 1, 2, ...$. Given the interest rate r_i between time i and $i+1$, the discounted value at time 0 of all those payments made till time t is

$$b()0 + b(1)(1+r_{t-1})^{-1} + b(2)(1+r_{t-1})^{-1}(1+r_{t-2})^{-1} + ... + b(t)(1+r_{t-1})^{-1}...(1+r_0)^{-1} \ .$$

This has exactly the same form as (2) and obeys the same equation.

The stochastic difference equation (1) is also obtained from the ARCH(1) (autoregressive-conditionally-heteroscedastic) models of log-returns:

$$R_{t+1} = \sqrt{b + aR_t^2}\ Z_t \ , \tag{10}$$

where Z_t is a gaussian random variable of zero mean and unit variance. This process (10) describes a persistence and thus clustering of volatilities R_t^2. Indeed, the factor $(b + aR_t^2)^{\frac{1}{2}}$ ensures that the amplitude of the motion R_{t+1} is controlled by the past realization of the amplitude R_t^2. Now, calling $X_t \equiv \langle R_t^2 \rangle$, where the average is carried

out over the realization of Z_t, it is clear that (10) is equivalent to (1), if we allow the coefficient a and b to depend on time (independently from Z_t).

In the context of insurance, X_t can be interpreted as the value of a perpetuity: the payments $b(t)$ are made at the beginning of each period and the accumulated payments X_{t-1} are subject to interest. The name perpetuity comes from "perpetual payment streams". See [17] for an introduction to perpetuities from a life insurance point of view and [18] from a pension fond point of view.

2.4 Immigration and investment portfolios

2.4.1 Expanding regime

Consider the case $\langle \log a \rangle > 0$ and define

$$r \equiv lim_{t\to\infty} \frac{1}{t} \langle \log b \rangle \ . \tag{11}$$

If $r = 0$, the additive term b becomes unimportant since X_t now diverges asymptotically and the process becomes purely multiplicative. We restrict our attention to the case where r is strictly positive. If $\langle \log a \rangle > r$, X_t diverges exponentially with log-normal fluctuations, since again X_t becomes asymptotically the result of a purely multiplicative process. A novel regime occurs when $\langle \log a \rangle < r$. In this case, let

$$b(t) = e^{r(t+1)} \hat{b}(t) \ , \tag{12}$$

where $\hat{b}(t)$ is a stochastic variable of order one. Let

$$a(t) = e^{r} \ \hat{a}(t) \ . \tag{13}$$

Since $r > \langle \log a \rangle$, then $\langle \log \hat{a} \rangle < 0$. Finally let

$$X_t = e^{rt} \hat{X}_t \ . \tag{14}$$

Equation (1) is then transformed into

$$\hat{X}_{t+1} = \hat{a} \hat{X}_t + \hat{b} \ , \tag{15}$$

where $\hat{a}$ and $\hat{b}$ obey the conditions of our previous analysis exactly. Thus $\hat{X}_t$ has a well-defined asymptotic non-singular pdf with a power law tail for large values.

The sequence X_t is not stationary since it blows up exponentially at the average rate r. But for a given t, one can characterize fully the *local* distribution of X over a small time interval from the asymptotic pdf of $\hat{X}_t$. Around the average exponential growth at rate r, X_t exhibits large fluctuations. Locally in time, the distribution of X_t is of the form

$$P(X_t) \sim e^{\mu r t} X_t^{-(1+\mu)} \ , \tag{16}$$

where μ is a solution of

$$\langle a^\mu \rangle = e^{r\mu} \ . \tag{17}$$

The simulation shown in Figure 3 shows an average exponential growth with superimposed fluctuations, much as in the case of Fig. 1.

There is a parallel situation to that considered above if $\langle \log a \rangle > 0$ with the constraint that X_t be smaller than some threshold $X_c > 0$. The pdf of X_t is then proportional to $X_t^{\mu-1}$ for $0 \leq X_t < X_c$, with μ given by equation (7) as before. This problem is the same as that studied by Grinstein et al. [19] for higher dimensional versions of the model.

2.4.2 Growth of immigrant populations

As an example of exponential growth, consider the problem of the growth of immigrant populations. Assume that the immigration flow b is increasing at a rate r. The national population may or may not grow exponentially with an average rate $\langle \log a \rangle$ which can be negative (as for Germany in the last quarter of century) or positive but with a growth rate smaller than that of the immigrant population. This condition holds in general for several occidental countries that are magnets for immigration from the developing countries. For example, the increased rate of flow b might be due to an average growth rate of population outside the country which is larger than the growth rate within the country itself, although we do not have to restrict ourselves to this case. From the analysis above, we conclude that, due to the influx of immigrants from poorer countries with rapidly growing populations, the growth rate of the host country ultimately approaches that of the poorer countries that are the sources of the influx. Our model predicts self-similar fluctuations of the host-country population around the average growth rate, due to the interplay between the natural fluctuations of the intrinsic growth rate of the host country and the immigrant flux. This model

assumes that immigrants will abandon the fertility rate they had previously and adjust it to that of the host country, on a time scale of the fluctuations of the intrinsic growth rate *. We have not tested these predictions for countries that have had a significant influx of immigrants. The simple model (1) is not expected to account for all growth histories but it might help to distinguish between different causes for the growth of population. We note also that economics Nobel laureate Simon had similar ideas to explain the growth laws for cities [20].

2.4.3 Growth of investment porfolios

Consider the growth of a portfolio of investments [21] such as that of a mutual fund. Consider a successful fund that exhibits a higher than market average return. Investors are thereby attracted to it and may invest new capital at a rate faster than the return rate itself. Not infrequently, new capital may flow in, even in the presence of relatively poor performance [22] †. Thus the condition $\langle \log a \rangle < \frac{1}{t} \langle \log b \rangle$ holds. The growth of the fund thus occurs not only through reinvestment of returns, but also through periodic additions under an investment plan. Due to the stochastic character of returns, the fund will show regular exponential growth, upon which is superimposed a series of spurts and gradual decays back to the exponential growth [21]; the fluctuations are distributed according to a power law. We have assumed that mutual funds are bought and sold as net asset value and their liquidity is linked with their growth. Since liquidity is commonly observed to be correlated with the volume of transactions, which is itself correlated with price volatility [23], these considerations might also be relevant to explain fat tails for price variation pdf's [24, 25].

Equation (1) may also describe the evolution of composite capital growth on the large scale of an economy [26], where $\{a = 1+$ return $-$ consumption$\}$ and b is the addition of new wealth from technological innovations or discoveries of mineral resources.

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*Indeed, at the next time step, the immigrant term b becomes part of the total population X which is updated with the (reduced) national growth rate a.

†This is a zeroth-order approximation since it can be expected that correlations exist between the return of a fund and the confidence of the investors.

2.5 The internet

There are probably many other areas where application of these ideas may prove to be fruitful. A recent paper by Huberman and Lukose [27] considers a multiplicative process to describe the population of users of the internet which cooperate. The model predicts intermittent congestions with short-lived spikes but lacks a random source of incoming users (the b term) which leads them to predict a log-normal distribution for Internet latenties. The reported data seems to exhibit a tail fatter than lognormal (see their figure 3), which we are tempted to interpret as the consequence of the multiplicative process, with intermittent amplification, coupled to a random additive source. In fact, Takayasu et al. [28] have found a power law pdf for the frequency of jams in Internet traffics.

3 Statistical properties

3.1 Generalization

Eq.(1) defines a stationary process if $\langle \ln b(t) \rangle < 0$. In order to get a power law pdf, $a(t)$ must sometimes take values larger than one, corresponding to intermittent amplifications. This is not enough: the presence of the additive term $b(t)$ (which can be constant or stochastic) is needed to ensure a "reinjection" to finite values, susceptible to the intermittent amplifications. It was thus shown [9] that (1) is only one among many convergent ($\langle \ln b(t) \rangle < 0$) multiplicative processes with repulsion from the origin (due to the $b(t)$ term in (1)):

$$X_{t+1} = e^{f(X_t, \{a,b,...\})} \; a \; X_t \; . \tag{18}$$

f has the following properties:

$$f(X_t, \{a, b, ...\}) \to 0 \quad \text{for} X_t \to \infty \; , \tag{19}$$

i.e. X_t is a pure multiplicative process when it is large,

$$f(X_t, \{a, b, ...\})) \to \infty \quad \text{for} X_t \to 0 \; , \tag{20}$$

i.e. X_t is repelled from 0. Additional conditions of monotonicity and regularity must be added to ensure that the pdf of X_t is unbound at large X_t.

All these processes share the same power law tail for their pdf $P(x) \sim x^{-1-\mu}$ for large x with μ solution of $\langle a(t)^{\mu} \rangle = 1$. $\ln x(t)$ undergoes a random walk with drift to the left which is repelled from $-\infty$. A simple Boltzmann argument shows that the stationary concentration profile is exponential, leading to the power law pdf in the $x(t)$ variable.

The deterministic version of equation (18) has been discussed frequently in the biological literature (see [2] and references therein). The model (1) is the special case $f(X_t, \{a(t), b(t), ...\}) = \log(1 + \frac{b(t)}{a(t)X_t})$. Thus the dynamics of X_t is the result of a multiplicative process that contracts on average. However the restocking term b prevents X_t from approaching zero by repulsion from the origin, and allows the distribution to converge to a non-trivial asymptotic pdf that turns out to be a power law under some mild assumptions stated below. In other words, the restocking term b which is a repulsion from the origin, corresponds to a resetting of the dynamics away from $X_t = 0$. The multiplicative process, with an a that can sometimes have values that are larger than 1, ensures an intermittent sensitive dependence on prior values of X_t. It is noteworthy that these ingredients of sensitive dependence on prior conditions and reinjection (of new stock), are also the two fundamental properties of systems that exhibit chaotic behavior [29].

3.2 Power law probability density function

Qualitatively, the existence of a limiting distribution for X_t that obeys (18), for a large class of $f(X, \{a, b, ...\})$ as in (19,20), is guaranteed by the competition between the convergence of X to zero and the sharp repulsion from it. Kesten's result for model (1) proves only the existence of a power law tail for the pdf but gives no information about the central part of the pdf, which is assumed at least to be an integral function. The most general mathematical results for this class of models are valid only for linear systems (1) and under a number of assumptions about the random coefficients

In the case of the generalized model, we can also give only a result for the tail of the pdf. With the additional condition $\partial f(X, \{a, b, ...\})/\partial x \to 0$ for $X \to \infty$, which is satisfied for a large subclass of smooth functions that satisfy conditions (19,20),

the pdf of X is the solution of

$$pdf\Big(Xe^{-f(X,\{\lambda,b,...\})}\Big) \;=\; pdf\Big(aX\Big) \quad . \tag{21}$$

The expression (21) means that the l.h.s. and r.h.s. have the same distribution. The solution to this problem has been given in [9] as a solution to a Wiener-Hopf equation for the tail. All the models (18) are characterized by a pdf with a tail that decays asymptotically for large X as a power law

$$P(X) \sim X^{-(1+\mu)} \quad , \tag{22}$$

if there is a solution $\mu > 0$ ‡ of the equation

$$\langle a^\mu \rangle = 1 \quad . \tag{23}$$

It has been shown in [30] that the exponent μ does not depend on the realizations of b if the distribution of a is smooth, and more generally, on the specific form of the repulsion from zero. For the example shown in the figures, the numerical solution to (23) is $\mu \approx 1.467$. The pdf $P(X)$ is always integrable at $X \to \infty$ because $\mu > 0$. Its integrability for small X is ensured automatically if b is bounded from below for $b > 0$ and can be shown to be true even when the pdf of b extends down to zero, provided it is not too singular at 0.

The result (22) with property (23) was proved first for model (1) by Kesten [1], and was then revisited by several authors in the differing contexts of finance [24] and 1D random-field Ising models [31]. Recently, Levy, Solomon and Ram [32] have rederived this result for a different submodel of the class (18) by the use of the extremal properties of the $G-$*harmonic* functions on non-compact groups [33], which are translational groups in this paper. In [9], a mapping to a biased random walk recovers these results. In [34], the method of characteristic functions has been used to recover the power law for the regime of infinite variance corresponding to $\mu \leq 2$.

As a consequence of the relatively short-range correlations of X_t and the properties of linear combinations of variables with power law pdf tails, the pdf of the variations $(X_{t+1} - X_t)$, and of higher order differences as well, also have a tail of the form (27) with the same exponent μ.

‡In general, when there is a real positive solution, there are also an infinitely discrete number of complex solutions (see below for their interpretation).

In the limit of small disorder on the multiplicative noise $\langle(\log a)^2\rangle - \langle\log a\rangle^2 \ll 1$, we can solve (23) by an expansion in $\langle a^\mu\rangle$ to second order in the cumulants of the pdf of a and get the approximation

$$\mu \approx \frac{|\langle\log a\rangle|}{\langle(\log a)^2\rangle - \langle\log a\rangle^2} \ . \tag{24}$$

This approximation is accurate for narrow pdf's and is exact if the pdf of $\log a$ is gaussian. For the case of the uniform distribution of the example, the estimate (24) is $\mu \approx 1.359$, which differs from the exact value 1.467 by about 7%.

3.3 Relation with auto-catalytic stochastic ODE

In the context of auto-catalytic equations that lead to multiplicative stochastic equations, Graham and Schenzle [35] have determined the exact asymptotic pdf for the variable X obeying the equation

$$\frac{dX}{dt} = -rX + pX^{1-p} + \eta X \quad , \tag{25}$$

where r is the decay rate in the absence of the last two terms and η is a multiplicative *Gaussian* noise with zero mean. If $p > 0$, then X^{1-p} is negligible for large X compared to rX but dominates as $X \to 0$. This terms plays exactly the same role as the function f in the discrete equation (18) by guaranteeing repulsion from $X = 0$. It is clear that a discretization of (25) yields (18) with $a(t) = 1 - r + \eta(t)$ [§]. In the case of Gaussian noise which is the only one that has been studied thus far, expression (22) is recovered, and the exponent is found again to be completely independent of the specifics of the repulsion from the origin, which is parameterized by the exponent $p > 0$ [¶] in this case. It is of interest to note that the discrete systems (1) and (18) with non-Gaussian noise lead to pdf's belonging to a *different universality class* than in the case of Gaussian noise, i.e. these systems will have different exponents for the same average and variance of the multiplicative noise. This property is due to the importance of rare large deviations on the determination of the exponent μ.

[§] Care must be taken to choose the correct representation (Ito or Stratonovich) and use the differential calculus accordingly in going from the discrete to the continuous description [36].

[¶] For $p < 0$, the repulsion is from infinity rather that from zero, and the dynamics is completely different.

3.4 Generalization to multi-dimensional processes

For completeness, there is a recent generalization of eq.(25) to multidimensional systems [19, 37],

$$\frac{\partial X}{\partial t} = D\frac{\partial^2 X}{\partial x^2} - rX + pX^{1-p} + \eta X \quad . \tag{26}$$

where $X(x,t)$ is now a field in a $d-$dimensional space x and D is a diffusion coefficient. Qualitatively, this equation describes a $d-$continuous infinity of variables X, each of which follows a multiplicative stochastic dynamics having the forms (18) or (25), and in the discretized equivalent is coupled to nearest neighbors through a diffusion term. The problems (25) and (18) correspond to the *zero*-dimensional case. Grinstein et al. [19] have studied the case $p < 0$ in an arbitrary number of dimensions, which leads to a repulsion from $+\infty$. Thus this case cannot lead to a pdf with a long tail and instead must belong to a completely different regime. Munoz and Hwa [37] have considered the case $p > 0$ which is relevant to our discussion, and also find a power law decay for the pdf of X. However, the determination of the exponent is done by numerical simulations and there are no exact results available for $d > 0$.

3.5 Complex exponents and log-periodicity

For a's with not-too-wide pdf's, complex solutions of (23) for the exponent μ can also be found. Complex values of μ lead to log-periodic oscillations that are superimposed on the lowest-order behavior which is power law [38, 39], as can be seen in the tail of figure 2. The underlying discrete scale invariance [40] has recently been found to be a widespread property of irreversible out-of-equilibrium processes [41, 42, 43, 44]. This log-periodicity overlaying the leading power law behavior of the probability density distribution is studied in details in [30] with emphasis on the progressive smoothing of the log-periodic structures as the randomness increases in order to test its robustness. It is shown that the log-periodicity is due to the intermittent amplifying multiplicative events.

3.6 Relation with intermittency in nonlinear dynamical systems

These results should not give the impression that we have a complete understanding of the systems (1) or (18). The results we have catalogued only describe the asymptotic tail for large X in the infinite time limit. The problem of the rate of convergence is much more involved and has only been addressed for eq. (25) in the case of Gaussian noise [35]. Furthermore, the apparent simple linearity of (1) masks equivalent nonlinear properties that are non-trivial. As mentioned, these systems have an intermittent sensitivity to both the prior values of X and the reinjection mechanism. The analogy with deterministic dynamical systems is even closer. Suppose that a is not truly stochastic but is itself generated by a deterministic dynamical equation, say $a(t) = \epsilon + r_t$, the latter given by the familiar logistic map $r_{t+1} = 4r_t(1 - r_t)$. The logistic map is well-known to be fully chaotic and has a invariant measure $P(r) = \frac{1}{\pi}[r(1-r)]^{-1/2}$. For positive but not too large ϵ, a obeys the condition $\langle \log a \rangle < 0$ while occasional individual realizations are larger than 1. Thus the fully deterministic coupled system,

$$X_{t+1} = (\epsilon + r_t)X_t + b \quad , \tag{27}$$

$$r_{t+1} = 4r_t(1 - r_t) \quad , \tag{28}$$

has the properties summarized above. We obtain the example exactly, if we replace (12) by the tent map

$$r_{t+1} = 2r_t \quad mod(2) \quad , \tag{29}$$

which is equivalent to (28) under a change of variable and has a uniform invariant measure in the interval $0 < r < 1$. The choice $\epsilon = 0.48$ gives the numerical example discussed in the introduction. The equivalence between the stochastic and deterministic chaotic processes for similar random walks has been checked in [45]. In our present problem, this result is correct as can be proved using the transition operator approach, which consists in describing the image of an arbitrary distribution density under the action of our random process. Ref.[30] then proves that the representation of the transition operator depends only on the stationary probabilities of the random couples (a, b), but not on the probabilities of the transitions between them, i.e. on the fact that the r_t are correlated.

3.7 Self-affinity and multiself-affinity

The process X_t defined by (1) or (18) is self-affine and can be characterized by its Hurst exponent H for $1 \leq \mu \leq 2$, through the expression [46]

$$\langle (X_{t+\Delta t} - X_t)^2 \rangle^{\frac{1}{2}} \sim (\Delta t)^H, \tag{30}$$

with

$$\frac{1}{2} \leq H = \frac{1}{\mu} \leq 1 \ , \tag{31}$$

For the parameters of figure 1, $H = 0.68$. The relation (30) is valid for $\Delta t > |t_c|$. To see this, the typical largest value X_{max} among Δt realizations of the variable X_t is given by

$$\Delta t \int_{X_{max}}^{+\infty} P(X) dX \sim 1 \quad , \tag{32}$$

which leads to

$$X_{max} \sim \Delta t^{1/\mu} \quad . \tag{33}$$

The second moment $\langle (X_{t+\Delta t} - X_t)^2 \rangle$ is then given by

$$\langle (X_{t+\Delta t} - X_t)^2 \rangle \sim \Delta t \int^{X_{max}} X^2 P(X) dX \sim \Delta t X_{max}^{2-\mu} \sim \Delta t^{2/\mu} \quad . \tag{34}$$

Equation (31) follows.

For a long sequence, the values X_t exhibit multi-self-affinity, as are log-normal distributions [47, 48]. It is also possible to add a space dimension to the model by constructing multiaffine fields so that the field obeys eq. (1) locally, and thus has all its properties. To do this, we use the construction given by Benzi, et al. [49] with a mother wavelet basis function which is a constant over the unit interval; these operations will be developed further elsewhere. The above properties hold for infinite time sequences. In the practical case of samples of finite length, finite size effects round off the power tails: for $X_t > e^{\sqrt{Dt}}$, (6) is approximately transformed into the log-normal law $\exp\left[-\frac{1}{2Dt}(\log X_t - Vt)^2\right]$, where $V \equiv \langle \log a \rangle$. This result arises because these large values of X_t are in the "free" multiplicative regime and are remote from the influence of the repulsive, additive terms b that are important near $X = 0$.

3.8 Extremes and durations of the intermittent bursts

It is useful to characterize further the nature of the intermittent bursts. It is possible to quantify the distribution of extremely large bursts. In this goal, we adapt the

result given by [50] for ARCH(1) processes and translate it to the case of (1). We thus obtain

$$lim_{t\to\infty} P\Big(max(X_1, X_2, ..., X_t) \leq x t^{1/\mu} \Big) = \exp[-c\theta x^{-\mu}] \ , \tag{35}$$

where the exponent μ is given by (23), the constant c is given by

$$c = \frac{\langle (b + aX_t)^\mu - (aX_t)^\mu \rangle}{\mu \langle a^\mu \ln a \rangle} \ , \tag{36}$$

and

$$\theta = \mu \int_1^\infty P\Big(max_{t\geq 1} \prod_{i=1}^t a(i) \leq \frac{1}{y} \Big) \ y^{-1-\mu} dy \ . \tag{37}$$

Expression (35) shows that the stationary process (1) have similar extremal properties as a sequence of independent and identically distributed random variables with the same probability density function. In the mathematical literature on extremes [50], θ is known as the "extremal index". It quantifies the role of dependence between the successive X_t in the realization of extremal values. In particular, $\theta < 1$ implies a smaller achieved extreme compared to the case of iid random variables with the same powerlaw distribution.

It is also possible to quantify the subset of times $1 \leq \{t_e\} \leq t$ at which X_{t_e} exceeds the threshold $x t^{\frac{1}{\mu}}$. In other words, among $X_1, X_2, ..., X_t$, some are above $x t^{\frac{1}{\mu}}$. What is the process describing these times of exceedance? It can be shown [50] that this subset converges in distribution to a compound Poisson process with intensity $c\theta x^{-\mu}$ and cluster probabilities

$$\pi_k = \frac{\theta_k - \theta_{k+1}}{\theta} \ , \tag{38}$$

where

$$\theta_k = \mu \int_1^\infty P\Big(\mathrm{card}\Big[\mathrm{t} / \prod_{\mathrm{i}=1}^{\mathrm{t}} \mathrm{a(i)} > \frac{1}{\mathrm{y}} \Big] = \mathrm{k} - 1 \Big) \mathrm{y}^{-1-\mu} \mathrm{dy} \ . \tag{39}$$

Note that $\theta_1 = \theta$ as defined by (37). $\mathrm{card}\Big[\mathrm{S}\Big]$ stands for the cardinal of the set S, i.e. the number of elements in that set.

Intuitively, the points of exceedance of a given threshold can be obtained in terms of the random walk $S_0 = 0$, $S_t = \sum_{i=1}^t \ln a(i)$ [50] (page 475). This is exact in the process studied in [32] defined by $X_{t+1} = \sup\{a(t) X_t, 1\}$, which defines a multiplicative process that is reflected from the left boundary $X = 1$. This reflection ensures the repulsion from the origin described as the generic mechanism for the generation

of power law pdf's by convergent multiplicative processes. Taking the logarithm and defining $x_t \equiv \ln X_t$, $\ln X_{t+1} = \sup\{\ln X_t + \ln a(t), 0\}$ which defines a random walk with a bias to the left ($\langle \ln a \rangle < 0$) and reflected at the origin. In between two successive reflections, this is a pure random walk. The process is thus the union of pieces of biased random walks, all starting from the origin and returning to it for the first time. At each collision with the origin, the random walk looses the memory of past motion. This correspondence allows us to get the exact distribution of durations between return to the origin, i.e. the distribution of the peak durations. Notice that this distribution $F(t,x)$ (which is the probability for the random walker to reach position x at time t, starting from the origin at time 0) also describes the width of a peak above an arbitrarily defined threshold, as the probability of return to the origin is independent of the definition of the origin. This distribution is a power law $\sim t^{-\frac{3}{2}}$ truncated for large t by the presence of the negative drift $v \equiv \langle \ln a \rangle$ which tends to bring back the random walk faster to the origin. The exact expression is obtained from the technique of generating functions [51] and reads (see [52] for an exact explicit relation and also [53])

$$F_v(t,x) = (1-v^2)^{\frac{t}{2}} \; e^{-\alpha x} F_{v=0}(t,x) \; , \tag{40}$$

where α is defined by $\cosh \alpha = \frac{1}{\sqrt{1-v^2}}$ (leading to $\alpha \to 0$ for $v \to 0$) and $F_{v=0}(t,x)$ is the probability of first passage at x starting from the origin at time 0, under no drift. The factor $(1-v^2)^{\frac{t}{2}} \; e^{-\alpha x}$ thus embodies the effect of the bias $v \equiv \langle \ln a \rangle < 0$.

The durations of intermittent amplifications are thus distributed according to $F_v(t,0)$ giving the pdf of the first return to the origin, starting from the origin at time 0. Since $F_{v=0}(t,0) \sim t^{-\frac{3}{2}}$, we get

$$F_v(t,0) \sim t^{-\frac{3}{2}} \; e^{\frac{\ln(1-v^2)}{2} \; t} \approx t^{-\frac{3}{2}} \; e^{-\frac{v^2}{2} \; t} \; , \tag{41}$$

where the last expression is valid for small drift. The same tail with a power law truncated by an exponential describes the other processes (18).

For the example of section 2, $v \equiv \langle \ln a \rangle = -0.06747$, leading to a characteristic time $|\frac{2}{\ln(1-v^2)}| \approx 440$. The pdf of the durations of intermittent amplifications is thus indistinguishable from the power law $t^{-\frac{3}{2}}$ for durations less than 440 and crosses over to an exponential tail for large durations. Figure 1 shows a sequence which is clearly in the first power law regime for the duration pdf.

4 A non-linear extension : optimization of restocking strategy

Up to now, the restocking term $b(t)$ has been considered to be independent of $X(t)$. As a first exercise, we now examine a case where $b(t)$ becomes a function of $X(t-1)$ so as to optimize a restocking strategy. This problem is motivated by the observation that, whether we consider the general problems of game management or the specific example of fish restocking, large fluctuations in population size from year to year are to be expected. One could thus hope to cushion the fluctuations by exerting a control on $b(t)$. In this spirit, variations of the model (1) have recently been proposed for the analysis of crop control in the presence of weed infestion [54], in which the control of restocking is done on an action on the multiplicative term. Here, we analyze the restocking strategies defined by the action of b, where b can become a function of X_t that do not modify the power-law property of the tail, for a broad class of functions $b(t, X_t)$ [‖], if the growth rate a is unknown and fluctuating. Indeed, large scale fluctuations are a robust feature of the intermittent multiplicative process (1) which do not depend on the specific nature of the reinjection mechanism at small $X(t)$. Optimized strategies $b(t)$ can be defined, even in the absence of correlations in the time series $a(t)$, if one knows the pdf $P_a(a)$.

We refer to [55] for an effort to model explicitly the dynamics of the fishermen behavior, coupled to the ecological and economic dynamics, which is partly based on neural nets. We also neglect nonlinear corrections between the annual number of offsprings and the size of the stock, which is important in absence of restocking [56] but is of minor concern when $\langle \log a \rangle < 0$ for which the restocking term is dominating the dynamics.

Suppose we want to prevent the population X_t from decreasing below some minimum X_{min}, and that any restocking action has a price, which can be taken to be a constant (overhead) plus a term that is proportional to the amount of added game, to a first approximation. The mathematical solution of this problem demands the

[‖] We note that the linear relation $b = \alpha X_t$ does not preserve the power-law property since this yields the purely multiplicative process $X_{t+1} = (a(t) + \alpha)X_t$. More generally, if b vanishes with X_t, the influence of restocking disappears for small X. It is the mechanism of repulsion from the origin induced by the b term that leads to the power law distribution.

definition of a cost function to be minimized. Many choices are possible and can be treated similarly. For the sake of illustration, let the cost function be proportional to the probability that X_t will be less than some X_{min}; if $X_t < X_{min}$ we declare that the year will be lost for fishermen or hunters and hence also for the various suppliers of their equipment. In term of economic loss, our simple model assumes that the price of such an event is some aggregate loss multiplied by the probability that it occurs. The price of restocking must be added to the price of the event and the sum minimized, which corresponds to a trade-off. We thus minimize $\int_0^{X_{min}} P(X_{t+1}|X_t, b(t))dX_{t+1} + \lambda b(t)$, with respect to $b(t)$ to find the best restocking strategy. The quantity $P(X_{t+1}|X_t, b(t))$ is the pdf of X_{t+1}, given the population X_t of the previous year and assuming $b(t)$ determined by the optimization. The factor λ is a measure of the price of restocking relative to the loss experienced when the population is too small. From (1), $P(X_{t+1}|X_t, b(t))$ is simply deduced from $P_a(a)$ by a change of variable. The solution of the minimization problem is

$$P_a(\frac{X_{min} - b(t)}{X_t}) = \lambda X_t. \qquad (42)$$

This equation has a non-zero solution for b only if λX_t is smaller than the maximum of P_a. If not, the solution is $b = 0$ which corresponds to the situation where either the previous population X_t was so large that next year is almost certain to be a good year without external action, or the price to restock λ is too large. If λX_t is smaller than the maximum of P_a, and P_a has a bell shape, there are two solutions to (42), but only the one which makes $\frac{X_{min}-b(t)}{X_t}$ the least is the genuine minimum of the cost function, the other being a maximum. The specific optimal strategy for choosing b thus depends on the detailed shape of the pdf P_a: this does not come as a surprise since the multiplicative process a is the dominant factor in the creation of stochasticity. We find, in agreement with intuition, that b is larger for smaller λX_t. In general, b is a non-linear function of X_t. However, since it goes to zero for large X_t, we conclude that the fluctuations for large X_t are not modified by the restocking strategy, which after all is only important for buffering the low values.

5 Concluding remarks

We have reviewed the main know properties of intermittent multiplicative processes. We have shown that they provide a robust and general model of intermittent self-similar dynamical processes with power law pdf tails. Even in the *complete absence* of non-linearity, the population X_t exhibits a non-trivial intermittent dynamics for $\langle \log a \rangle < \frac{1}{t}\langle \log b \rangle$, whether b be determined deterministically or stochastically. If b is bounded or grows sub-exponentially, then $\frac{1}{t}\langle \log b \rangle \to 0$ and the multiplicative process has to be convergent ($\langle \log a \rangle < 0$). The competition between the random multiplicative process a and the external source b is enough to produce intermittency as measured by the Hurst exponent and power law distributions. These results also hold for the more general class of models (18). The power law distributions are the hallmark of convergent multiplicative processes repelled from the origin and do not need the fine tuning of a control parameter as in usual models of criticality [57]. The model (1) provides a simple useful alternative to self-organizing models with nonlinear interactions [58, 59]. We have however shown how these stochastic linear systems can appear to have the properties of a nonlinear dynamical system.

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